# Equations of First order and First degree

## Linear Equation

The most general linear equation of first order can be written ~~as~~ in the form

$$\frac{dy}{dx} + Py = Q \quad \text{——— (1)}$$

Where P & Q are either functions of $x$ or constants (including zero).

If both P and Q are both constants, then the variables are separable; the same is true if either, P or Q is zero.

(I) Let $Q = 0$, $P \neq 0$. Then

$$\frac{dy}{dx} + Py = 0 \quad \text{——— (2)}$$

$$\Rightarrow \frac{dy}{y} = -P\,dx$$

integrating, we get

$$\log y = -\int P\,dx$$

$$\Rightarrow y = ce^{-\int P\,dx}$$

Where c is the constant of integration.

(II) When $Q \neq 0$ & $P = 0$

$$\Rightarrow \frac{dy}{dx} = Q \quad \Rightarrow dy = Q\,dx \Rightarrow y = c + \int Q\,dx$$

If $P = Q = 0$, then $y = c$.

(III) When $Q \neq 0$

We assume that $y = v e^{-\int P dx}$ is a solution of equation (1), where $Q \neq 0$

Then $\frac{dy}{dx} = \frac{dv}{dx} e^{-\int P dx} - Pv e^{-\int P dx}$ —— (3)

putting the value of $\frac{dy}{dx}$ in (1), we get

$$\frac{dy}{dx} + Py = Q$$

$$\Rightarrow \frac{dv}{dx} e^{-\int P dx} - Pv e^{-\int P dx} + Pv e^{-\int P dx} = Q$$

$$\Rightarrow \frac{dv}{dx} = Q e^{\int P dx}$$

$$\Rightarrow dv = Q e^{\int P dx} dx$$

Integrating, we have

$$\int dv = \int \{Q e^{\int P dx}\} dx$$

$$\Rightarrow v = \int \{Q e^{\int P dx}\} dx + c$$

Where $c$ is the arbitrary constant of integration.

$$\therefore y = v\, e^{-\int P dx}$$

$$= e^{-\int P dx}\left[c + \int \{Q e^{\int P dx}\} dx\right]$$

$$\therefore y = c e^{-\int P dx} + e^{-\int P dx} \int \{Q \cdot e^{\int P dx}\} dx$$

Ans.

(IV) Integrating factor method :→

(1) $e^{\int P dx}$ is an integrating factor of the equation.

(2) Multiply both sides by $e^{\int P dx}$. The equation becomes

$$\frac{dy}{dx} e^{\int P dx} + P y\, e^{\int P dx} = Q e^{\int P dx}$$

$$\Rightarrow \frac{d}{dx}\{y \cdot e^{\int P dx}\} = Q \cdot e^{\int P dx}$$

Integrating, we have

$$y \cdot e^{\int P dx} = \int \{Q \cdot e^{\int P dx}\} dx + c$$

$\therefore$ The general solution is:

$$y = e^{-\int P dx} \left\{ c + \int \{Q e^{\int P dx}\} dx \right\}$$

Ans.

**Example ①** Find the solution of the differential Equation

$$\frac{dy}{dx} + 2xy = 0$$

in the form $y = y_1$. ~~Hen~~ Hence solve

$\frac{dy}{dx} + 2xy = 4x$ by the substitution $y = vy_1$.

**Solution:** → Now,

$$\frac{dy}{dx} + 2xy = 0 \Rightarrow \frac{dy}{y} + 2x\,dx = 0$$

∴ Integrating

$$\log y - \log c_1 + x^2 = 0$$

$$\Rightarrow \log\frac{y}{c_1} = -x^2 \quad \text{or,} \quad y = c_1 e^{-x^2}$$

We take $y = y_1 = e^{-x^2}$ which is a solution of the reduced equation $\frac{dy}{dx} + 2xy = 0$ disregarding the constant of integration

Let $y = vy_1 = ve^{-x^2}$

$$\therefore \frac{dy}{dx} = \frac{dv}{dx}e^{-x^2} - 2xve^{-x^2}$$

putting in $\frac{dy}{dx} + 2xy = 4x$, we have

$$\frac{dv}{dx}e^{-x^2} - 2xve^{-x^2} + 2xve^{-x^2} = 4x$$

$$\Rightarrow \frac{dv}{dx} = 4xe^{x^2}$$

Integrating

$$v = \int 4xe^{x^2}dx = 2e^{x^2} + c$$

∴ General solution is

$$y = vy_1 = e^{-x^2}(2e^{x^2} + c) = 2 + ce^{-x^2}$$

Ans.

Example ② Solve by the method of variation of Parameters

$$\frac{dy}{dx} - 5y = \sin x. \quad \text{— ①}$$

Solution :→ The reduced equation is

$$\frac{dy}{dx} - 5y = 0 \Rightarrow \frac{dy}{y} = 5dx$$

Integrating

$$\log y = 5x + \log c_1$$

$$\Rightarrow \log \frac{y}{c_1} = 5x \quad \Rightarrow y = c_1 e^{5x}$$

Disregarding the constant of integration, we take $y = y_1 = e^{5x}$ as one of the solutions of the reduced equation.

Take $y = vy_1 = ve^{5x}$

$$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} e^{5x} + 5ve^{5x}$$

$$\because \frac{dy}{dx} - 5y = \sin x$$

$$\Rightarrow \frac{dv}{dx} e^{5x} + 5ve^{5x} - 5ve^{5x} = \sin x$$

$$\Rightarrow e^{5x} \frac{dv}{dx} = \sin x$$

$$\Rightarrow v = \int e^{-5x} \sin x \, dx = e^{-5x}\left(-\frac{5}{26} \sin x - \frac{1}{26} \cos x\right) + c$$

∴ General solution is

$$y = ve^{5x} = ce^{5x} - \frac{5}{26} \sin x - \frac{1}{26} \cos x$$

Ans.

(9) Solve $y^2 + \left(x - \frac{1}{y}\right)\frac{dy}{dx} = 0$

Solution: → Since the equation contains $y^2$ & so it cannot be linear in $y$.

We try to see weither it is linear in $x$.

$$\frac{xy-1}{y}\frac{dy}{dx} = -y^2$$

$$\Rightarrow -\left(\frac{xy-1}{y^3}\right) = \frac{dx}{dy}$$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{y^2}x = \frac{1}{y^3} \qquad \text{--- (1)}$$

Which is linear in $x$

$$\therefore \text{I.F.} = e^{\int \frac{1}{y^2}dy} = e^{-\frac{1}{y}}$$

Multiplying equation (1) by I.F. we have

$$\frac{d}{dy}\left\{x e^{-\frac{1}{y}}\right\} = e^{-\frac{1}{y}} \cdot \frac{1}{y^3}$$

integrating

$$x e^{-\frac{1}{y}} = e^{-\frac{1}{y}}\left(1 + \frac{1}{y}\right) + c$$

or, $x = 1 + \frac{1}{y} + c e^{\frac{1}{y}}$

Ans.

(Q) Solve $\frac{dy}{dx} \cancel{x} + \frac{y}{x} = x^3$

Solution:-

Here $P = \frac{1}{x}$ , $Q = x^3$

$\therefore$ I.F. $= e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$

Hence the general solution is

$$\frac{d}{dx}\{y \cdot (\text{I.F.})\} = \int Q \cdot (\text{I.F.})$$

$$\Rightarrow y \cdot x = \int x^3 \cdot x \, dx$$

$$\Rightarrow yx = \frac{x^5}{5} + c$$

Ans.